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CITATION:

Miyachi, Jun-ichi. Derived Categories in Representation Theory (Expansion of Lie Theory and New Advances). 数理解析研究所講究録 2003, 1348: 101-117

ISSUE DATE:

2003-12

URL:

<http://hdl.handle.net/2433/25098>

RIGHT:

# Derived Categories in Representation Theory

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We survey recent methods of derived categories in the representaion theory of algebras.

## 1 Triangulated Categories and Brown Representability

**Definition 1.1** *A triangulated category  $\mathcal{C}$  is an additive category together with*

- (1) *an autofunctor  $T : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$  (i.e. there is  $T^{-1}$  such that  $T \circ T^{-1} = T^{-1} \circ T = 1_{\mathcal{C}}$ ) called the translation, and*
- (2) *a collection  $\mathcal{T}$  of sextuples  $(X, Y, Z, u, v, w)$ :*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

*called (distinguished) triangles. These data are subject to the following four axioms:*

- (TR1) (1) *Every sextuple  $(X, Y, Z, u, v, w)$  which is isomorphic to a (distinguished) triangle is a (distinguished) triangle.*
- (2) *Every morphism  $u : X \rightarrow Y$  is embedded in a (distinguished) triangle*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

$$\begin{array}{ccc} & Z & \\ w \swarrow & & \nwarrow v \\ (1) \quad X & \xrightarrow{u} & Y \end{array}$$

- (3) *For any  $X \in \mathcal{C}$ ,*

$$X \xrightarrow{1} X \rightarrow 0 \rightarrow T(X)$$

*is a (distinguished) triangle*

(TR2) A sextuple

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is a (distinguished) triangle if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$$

is a (distinguished) triangle.

(TR3) For any (distinguished) triangles  $(X, Y, Z, u, v, w)$ ,  $(X', Y', Z', u', v', w')$  and a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \xrightarrow{w} T(X) \\ \downarrow f & & \downarrow g & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \xrightarrow{w'} T(X') \end{array}$$

there exists  $h : Z \rightarrow Z'$  which makes a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \xrightarrow{w} T(X) \\ \downarrow f & & \downarrow g & & \downarrow h \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' \xrightarrow{w'} T(X') \end{array}$$

(TR4) (Octahedral axiom) For any two consecutive morphisms  $u : X \rightarrow Y$  and  $v : Y \rightarrow Z$ , if we embed  $u$ ,  $vu$  and  $v$  in (distinguished) triangles  $(X, Y, Z', u, i, i')$ ,  $(X, Z, Y', vu, k, k')$  and  $(Y, Z, X', v, j, j')$ , respectively, then there exist morphisms  $f : Z' \rightarrow Y'$ ,  $g : Y' \rightarrow X'$  such that the following diagram commute

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & T(X) \\ \parallel & & \downarrow v & & \downarrow f & & \parallel \\ X & \xrightarrow{vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & T(X) \\ & & \downarrow j & & \downarrow g & & \downarrow T(u) \\ & & X' & \xrightarrow{j'} & X' & \xrightarrow{j'} & TY \\ & & \downarrow j' & & \downarrow T(i)j' & & \\ & & T(Y) & \xrightarrow{T(i)} & T(Z') & & \end{array}$$

and the third column is a triangle.

Sometimes, we write  $X[i]$  for  $T^i(X)$ .

**Definition 1.2 ( $\partial$ -functor)** Let  $\mathcal{C}, \mathcal{C}'$  be triangulated categories. An additive functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  is called a  $\partial$ -functor (sometimes exact functor) provided that there is a functorial isomorphism  $\alpha : FT_{\mathcal{C}} \xrightarrow{\sim} T_{\mathcal{C}'}F$  such that

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\alpha_X F(w)} T_{\mathcal{C}'}(F(X))$$

is a triangle in  $\mathcal{C}'$  whenever  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T_{\mathcal{C}}(X)$  is a triangle in  $\mathcal{C}$ . Moreover, if a  $\partial$ -functor  $F$  is an equivalence, then  $F$  is called a triangulated equivalence. In this case, we denote by  $\mathcal{C} \stackrel{\Delta}{\cong} \mathcal{C}'$ .

For  $(F, \alpha), (G, \beta) : \mathcal{C} \rightarrow \mathcal{C}'$   $\partial$ -functors, a functorial morphism  $\phi : F \rightarrow G$  is called a  $\partial$ -functorial morphism if

$$(T_{\mathcal{C}'}\phi) \circ \alpha = \beta \circ \phi T_{\mathcal{C}}$$

We denote by  $\partial(\mathcal{C}, \mathcal{C}')$  the collection of all  $\partial$ -functors from  $\mathcal{C}$  to  $\mathcal{C}'$ , and denote by  $\partial\text{Mor}(F, G)$  the collection of  $\partial$ -functorial morphisms from  $F$  to  $G$ .

**Proposition 1.3** Let  $F : \mathcal{C} \rightarrow \mathcal{C}'$  be a  $\partial$ -functor between triangulated categories. If  $G : \mathcal{C}' \rightarrow \mathcal{C}$  is a right (or left) adjoint of  $F$ , then  $G$  is also a  $\partial$ -functor.

**Definition 1.4** A contravariant (resp., covariant) additive functor  $H : \mathcal{C} \rightarrow \mathcal{A}$  from a triangulated category  $\mathcal{C}$  to an abelian category  $\mathcal{A}$  is called a homological functor (resp., a cohomological functor), if for any triangle  $(X, Y, Z, u, v, w)$  in  $\mathcal{C}$  the sequence

$$\begin{aligned} H(T(X)) \rightarrow H(Z) \rightarrow H(Y) \rightarrow H(X) \\ (\text{resp., } H(X) \rightarrow H(Y) \rightarrow H(Z) \rightarrow H(T(X))) \end{aligned}$$

is exact. Taking  $H(T^i(X)) = H^i(X)$ , we have the long exact sequence:

$$\begin{aligned} \dots \rightarrow H^{i+1}(X) \rightarrow H^i(Z) \rightarrow H^i(Y) \rightarrow H^i(X) \rightarrow \dots \\ (\text{resp., } \dots \rightarrow H^i(X) \rightarrow H^i(Y) \rightarrow H^i(Z) \rightarrow H^{i+1}(X) \rightarrow \dots) \end{aligned}$$

**Proposition 1.5** The following hold.

1. If  $(X, Y, Z, u, v, w)$  is a triangle, then  $vu = 0$ ,  $wv = 0$  and  $T(u)w = 0$ .
2. For any  $X \in \mathcal{C}$ ,  $\text{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \rightarrow \mathcal{A}\mathfrak{b}$  (resp.,  $\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \mathcal{A}\mathfrak{b}$ ) is a homological functor (resp., a cohomological functor).
3. For any homomorphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow T(f) \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & T(X') \end{array}$$

if two of  $f$ ,  $g$  and  $h$  are isomorphisms, then the rest is also an isomorphism.

**Definition 1.6 (Compact Object)** Let  $\mathcal{C}$  be a triangulated category. An object  $C \in \mathcal{C}$  is called a compact object in  $\mathcal{C}$  if the canonical morphism

$$\coprod_{i \in I} \text{Hom}_{\mathcal{C}}(C, X_i) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(C, \coprod_{i \in I} X_i)$$

is an isomorphism for any set  $\{X_i\}_{i \in I}$  of objects (if  $\coprod_{i \in I} X_i$  exists in  $\mathcal{C}$ ).

For a triangulated category  $\mathcal{C}$ , a set  $\mathcal{S}$  of compact objects is called a generating set if  $\text{Hom}_{\mathcal{C}}(\mathcal{S}, X) = 0 \Rightarrow X = 0$ , and if  $T(\mathcal{S}) = \mathcal{S}$ . A triangulated category  $\mathcal{C}$  is compactly generated if  $\mathcal{C}$  contains arbitrary coproducts, and if it has a generating set.

**Definition 1.7 (Homotopy Limit)** Let  $\mathcal{C}$  be a triangulated category which contains arbitrary coproducts (resp., products). For a sequence  $\{X_i \rightarrow X_{i+1}\}_{i \in \mathbb{N}}$  (resp.,  $\{X_{i+1} \rightarrow X_i\}_{i \in \mathbb{N}}$ ) of morphisms in  $\mathcal{C}$ , the homotopy colimit (resp., homotopy limit) of the sequence is the third (resp., second) term of the triangle

$$\coprod_i X_i \xrightarrow{1-\text{shift}} \coprod_i X_i \rightarrow \underline{\text{hocolim}} X_i \rightarrow T\left(\coprod_i X_i\right)$$

$$(\text{resp.}, T^{-1}\left(\prod_i X_i\right) \rightarrow \underline{\text{holim}} X_i \rightarrow \prod_i X_i \xrightarrow{1-\text{shift}} \prod_i X_i)$$

where the above shift morphism is the coproduct (resp., product) of  $X_i \xrightarrow{f_i} X_{i+1}$  (resp.,  $X_{i+1} \xrightarrow{f_i} X_i$ ) ( $i \in \mathbb{N}$ ).

**Proposition 1.8** Let  $\mathcal{C}$  be a triangulated category which contains arbitrary coproducts,  $\{X_i \rightarrow X_{i+1}\}_{i \in \mathbb{N}}$  a sequence of morphisms in  $\mathcal{C}$ . For a compact object  $C$  in  $\mathcal{C}$ , we have

$$\text{Hom}(C, \underline{\text{hocolim}} X_i) \cong \varinjlim \text{Hom}(C, X_i)$$

*Proof.* We have an exact sequence

$$0 \rightarrow \coprod_i \text{Hom}(C, X_i) \rightarrow \coprod_i \text{Hom}(C, X_i) \rightarrow \text{Hom}(C, \underline{\text{hocolim}} X_i) \rightarrow 0 \quad \square$$

**Theorem 1.9 (Brown Representability Theorem [Ne])** Let  $\mathcal{C}$  be a compactly generated triangulated category. If a homological functor  $H : \mathcal{C} \rightarrow \mathcal{A}b$  sends coproducts to products, then it is representable, that is, there is an object  $X \in \mathcal{C}$  such that  $H \cong \text{Hom}_{\mathcal{C}}(-, X)$ .

*Sketch of Proof.* Let  $\mathcal{S}$  be a generating set of  $\mathcal{C}$ . There exist a coproduct  $X_1$  of objects of  $\mathcal{S}$  and a morphism  $h_{X_1} \rightarrow H$  such that  $\text{Hom}_{\mathcal{C}}(C, X_1) \rightarrow H(C)$  is surjective for any  $C \in \mathcal{S}$ . For a functor  $K_1 = \text{Ker}(h_{X_1} \rightarrow H)$  there exists a coproduct  $Z_2$  of objects in  $\mathcal{S}$  and a morphism  $h_{Z_2} \rightarrow K_1$  such that  $\text{Hom}_{\mathcal{C}}(C, Z_2) \rightarrow K_1(C)$  is surjective for any  $C \in \mathcal{S}$ . Then we have a triangle  $Z_2 \rightarrow X_1 \rightarrow X_2 \rightarrow Z_2[1]$ . Since  $H$  is a homological functor, we have a commutative diagram

$$\begin{array}{ccccc} H(X_2) & \longrightarrow & H(X_1) & \longrightarrow & H(Z_2) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \text{Mor}(h_{X_2}, H) & \longrightarrow & \text{Mor}(h_{X_1}, H) & \longrightarrow & \text{Mor}(h_{Z_2}, H) \end{array}$$

Then there is a morphism  $X_1 \rightarrow X_2$  satisfying a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & \text{Hom}_{\mathcal{C}}(-, X_1) & \longrightarrow & H \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & K_2 & \longrightarrow & \text{Hom}_{\mathcal{C}}(-, X_2) & \longrightarrow & H \end{array}$$

and we have a morphism of exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1(C') & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, X_1) & \longrightarrow & H(C) \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow & & \parallel \\ 0 & \longrightarrow & K_2(C') & \longrightarrow & \text{Hom}_{\mathcal{C}}(C, X_2) & \longrightarrow & H(C) \longrightarrow 0 \end{array}$$

for any  $C \in \mathcal{S}$ . By inductive step, we have a triangle

$$\coprod_i X_i \xrightarrow{1-\text{shift}} \coprod_i X_i \rightarrow \text{hocolim } X_i \rightarrow T\left(\coprod_i X_i\right)$$

and we have an exact sequence

$$\begin{array}{ccccc} H(\text{hocolim } X_i) & \longrightarrow & \prod_i H(X_i) & \longrightarrow & \prod_i H(X_i) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \text{Mor}(\text{h}_{\text{hocolim } X_i}, H) & \longrightarrow & \prod_i \text{Mor}(\text{h}_{X_i}, H) & \longrightarrow & \prod_i \text{Mor}(\text{h}_{X_i}, H) \end{array}$$

Therefore there is a morphism  $\text{Hom}_{\mathcal{C}}(-, \text{hocolim } X_i) \rightarrow H$  such that

$$\text{Hom}_{\mathcal{C}}(C, \text{hocolim } X_i) \cong H(C)$$

for any  $C \in \mathcal{S}$ . Hence we have  $\text{Hom}_{\mathcal{C}}(-, \text{hocolim } X_i) \cong H$ .  $\square$

**Corollary 1.10 (Adjoint Functor Theorem [Ne])** *Let  $\mathcal{C}$  be a compactly generated triangulated category. If a  $\partial$ -functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  commutes with arbitrary coproducts, then there exists a  $\partial$ -functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  which is a right adjoint of  $F$ .*

*Proof.* For any  $Y \in \mathcal{D}$ , the functor

$$\text{Hom}_{\mathcal{D}}(F(-), Y) : \mathcal{C} \rightarrow \mathfrak{Ab}$$

is a homological functor. By Brown representability theorem there is an object  $GY \in \mathcal{C}$  such that

$$\text{Hom}_{\mathcal{D}}(F(-), Y) \cong \text{Hom}_{\mathcal{C}}(-, GY) \quad \square$$

**Definition 1.11 (Multiplicative System)** *Let  $S$  be a multiplicative system in a triangulated category  $\mathcal{C}$  which satisfies the following conditions:*

- (FR0) *For a morphism  $s$  in  $\mathcal{C}$ , if there exist  $f, g$  such that  $sf, gs \in S$ , then  $s \in S$ .*
- (FR1) *(1)  $1_X \in S$  for every  $X \in \mathcal{C}$ .  
(2) For  $s, t \in S$ , if  $st$  is defined, then  $st \in S$ .*

(FR2) Every diagram in  $\mathcal{C}$

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ f \downarrow & & \\ X' & & \end{array}$$

with  $s \in S$ , can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ f \downarrow & & \downarrow g \\ X' & \xrightarrow{t} & Y' \end{array}$$

with  $s, t \in S$ . Ditto for the statement with all arrows reversed.

(FR3) For  $f, g \in \text{Hom}_{\mathcal{C}}(X, Y)$  the following are equivalent.

- (1) There exists  $s \in S$  such that  $sf = sg$ .
- (2) There exists  $t \in S$  such that  $ft = gt$ .

(FR4) For a morphism  $u$  in  $\mathcal{C}$ ,  $u \in S$  if and only if  $T(u) \in S$ .

(FR5) For triangles  $(X, Y, Z, u, v, w)$ ,  $(X', Y', Z', u', v', w')$  and morphisms  $f : X \rightarrow X'$ ,  $g : Y \rightarrow Y'$  in  $S$  with  $gu = u'f$ , there exists  $h : Z \rightarrow Z'$  in  $S$  such that  $(f, g, h)$  is a homomorphism of triangles.

**Definition 1.12 (Quotient Category)** We define the quotient category  $S^{-1}\mathcal{C}$  of  $\mathcal{C}$ , as follows:

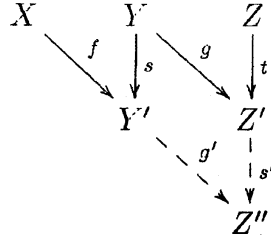
1.  $\text{Ob}(S^{-1}\mathcal{C}) = \text{Ob}(\mathcal{C})$ .
2. For  $X, Y \in \text{Ob}(\mathcal{C})$ , let  $V(X, Y) = \{(s, Y', f) \mid s : Y \rightarrow Y' \in S, f : X \rightarrow Y'\}$ . In  $V(X, Y)$ , we define  $(s, Y', f) \sim (s', Y'', f')$  if there is  $(s'', Y''', f'')$  such that all triangles are commutative in the following diagram:

$$\begin{array}{ccccc} & & Y' & & \\ & f \nearrow & \downarrow & \nwarrow s & \\ X & \xrightarrow{f''} & Y''' & \xleftarrow{s''} & Y \\ & f' \searrow & \uparrow & \swarrow s' & \\ & & Y'' & & \end{array}$$

Then we define a morphism from  $X$  to  $Y$  by an equivalence class  $s^{-1}f$  of  $(s, Y', f)$ .

3. For  $s^{-1}f : X \rightarrow Y, t^{-1}g : Y \rightarrow Z$ , by (FR2) there are  $s' : Z' \rightarrow Z'' \in S$  and

$g' : Y' \rightarrow Z''$  such that  $s'og = g'os$ . Then we define  $(t^{-1}g) \circ (s^{-1}f) = (s'ot)^{-1}g \circ f$ .



Moreover, we define the quotient functor  $Q : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$  by

(Q1)  $Q(X) = X$  for  $X \in \mathcal{C}$ .

(Q2)  $Q(f) = 1_Y^{-1}f$  for a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ .

**Remark 1.13** Can we define (2) in the above?

**Definition 1.14 (Épaisse Subcategory)** Let  $\mathcal{C}$  be a triangulated category. A full subcategory  $\mathcal{U}$  of  $\mathcal{C}$  is called a full triangulated subcategory if  $X \rightarrow Y$  is a morphism in  $\mathcal{U}$ , then there is a triangle  $X \rightarrow Y \rightarrow Z \rightarrow TX$  with  $Z \in \mathcal{U}$ .

A full triangulated subcategory  $\mathcal{U}$  is called an épaisse subcategory if it is closed under direct summands. In this case, let  $S(\mathcal{U})$  be the collection of morphisms  $s$  such that  $X \xrightarrow{s} Y \rightarrow Z \rightarrow X[1]$  is a triangle with  $Z \in \mathcal{U}$ . Then  $S(\mathcal{U})$  is a multiplicative system satisfying (FR0) - (FR5). We write  $\mathcal{C}/\mathcal{U} = S(\mathcal{U})^{-1}\mathcal{C}$ .

In the case that  $\mathcal{C}$  contains arbitrary coproducts, a full triangulated subcategory  $\mathcal{U}$  is called a localizing subcategory if it is closed under coproducts.

**Remark 1.15** The above definition of an épaisse subcategory  $\mathcal{U}$  is the same as the original definition [Ve], that is, a full triangulated category satisfying that if  $X \xrightarrow{u} Y$  factors through some object in  $\mathcal{U}$  and if there is a triangle  $X \xrightarrow{u} Y \rightarrow Z \rightarrow T(X)$  with  $Z \in \mathcal{U}$ , then  $X, Y \in \mathcal{U}$ .

**Proposition 1.16 ([BN])** Let  $\mathcal{C}$  be a triangulated category which contains arbitrary coproducts. Then any localizing subcategory is an épaisse subcategory.

*Sketch of Proof.* Let  $\mathcal{U}$  be a localizing subcategory, and  $X \in \mathcal{U}$  with  $X = Y \oplus Z$  in  $\mathcal{C}$ . We take a morphism  $e : X \rightarrow Y \hookrightarrow X$ , and consider the sequence of morphisms

$$(*) \quad X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} \dots$$

Then it is easy to see that  $Y \cong \varinjlim (*) \in \mathcal{U}$ . □

**Proposition 1.17** Let  $\mathcal{C}$  be a triangulated category. For a multiplicative system  $S$  satisfying the conditions (FR0) - (FR5), let  $\mathcal{U}(S)$  be the full triangulated subcategory consisting of objects  $Z$  which is in a triangle  $X \xrightarrow{s} Y \rightarrow Z \rightarrow X[1]$  with  $s \in S$ . Then the following hold.



1.  $S(\mathcal{U})$  and  $\mathcal{U}(S)$  induce an 1 - 1 correspondence between the collection of multiplicative systems  $S$  satisfying the conditions (FR0) - (FR5) and the collection of épaisse subcategories  $\mathcal{U}$ .
2. For an épaisse subcategory  $\mathcal{U}$ ,  $\mathcal{C}/\mathcal{U}$  is a triangulated category whose (distinguished) triangles are defined to be isomorphic to (distinguished) triangles of  $\mathcal{C}$ .
3. Assume  $\mathcal{C}$  contains arbitrary coproducts. For a localizing subcategory  $\mathcal{U}$ ,  $\mathcal{C}/\mathcal{U}$  also contains arbitrary coproducts.

**Definition 1.18 (stable  $t$ -structure)** For full subcategories  $\mathcal{U}$  and  $\mathcal{V}$  of a triangulated category  $\mathcal{C}$ ,  $(\mathcal{U}, \mathcal{V})$  is called a stable  $t$ -structure in  $\mathcal{C}$  provided that

1.  $\mathcal{U}$  and  $\mathcal{V}$  are stable for translations.
2.  $\text{Hom}_{\mathcal{C}}(\mathcal{U}, \mathcal{V}) = 0$ .
3. For every  $X \in \mathcal{C}$ , there exists a triangle  $U \rightarrow X \rightarrow V \rightarrow TU$  with  $U \in \mathcal{U}$  and  $V \in \mathcal{V}$ .

**Proposition 1.19 ([BBD], c.f. [Mi])** Let  $\mathcal{C}$  be a triangulated category,  $(\mathcal{U}, \mathcal{V})$  a stable  $t$ -structure in  $\mathcal{C}$ , and  $i_* : \mathcal{U} \rightarrow \mathcal{C}, j_* : \mathcal{V} \rightarrow \mathcal{C}$  the canonical embeddings. Then the following hold.

1.  $\mathcal{U}$  and  $\mathcal{V}$  is épaisse subcategories of  $\mathcal{C}$ .
2.  $i_*$  (resp.,  $j_*$ ) has a right adjoint  $i^!$  (resp., a left adjoint  $j^*$ ).
3. The adjunction arrows induce a triangle

$$i_* i^! X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} j_* j^* X \rightarrow i_* i^! X[1]$$

for any  $X \in \mathcal{C}$ .

4.  $\mathcal{C}/\mathcal{U}$  (resp.,  $\mathcal{C}/\mathcal{V}$ ) exists, and it is triangulated equivalent to  $\mathcal{V}$  (resp.,  $\mathcal{U}$ ).

$$\begin{array}{ccccc}
 & \mathcal{C}/\mathcal{V} & & & \\
 & \uparrow i & \swarrow & & \\
 \mathcal{U} & \xrightarrow{i_*} & \mathcal{C} & \xrightarrow{j_*} & \mathcal{V} \\
 & \xleftarrow{i^!} & & \xleftarrow{j^*} & \\
 & & \searrow & & \\
 & & \mathcal{C}/\mathcal{U} & & 
 \end{array}$$

**Corollary 1.20** Let  $\mathcal{C}$  be a compactly generated triangulated category, and  $\mathcal{U}$  a localizing subcategory of  $\mathcal{C}$ . Then  $\mathcal{C}/\mathcal{U}$  can be defined if and only if there is a full triangulated subcategory  $\mathcal{V}$  such that  $(\mathcal{U}, \mathcal{V})$  a stable  $t$ -structure in  $\mathcal{C}$ .

*Proof.* If  $\mathcal{C}/\mathcal{U}$  can be defined, then the quotient functor  $Q : \mathcal{C} \rightarrow \mathcal{C}/\mathcal{U}$  commutes with coproducts. By Adjoint Functor Theorem,  $Q$  has a right adjoint  $F : \mathcal{C}/\mathcal{U} \rightarrow \mathcal{C}$ . By Proposition 1.19, it is easy to see that  $(\mathcal{U}, \text{Im } F)$  is a stable  $t$ -structure in  $\mathcal{C}$ .  $\square$

## 2 Derived Categories

Throughout this section,  $\mathcal{A}$  is an abelian category and  $\mathcal{B}$  is an additive subcategory of  $\mathcal{A}$  which is closed under isomorphisms.

**Definition 2.1 (Complex)** A (cochain) complex is a collection  $X^\cdot = (X^n, d_X^n : X^n \rightarrow X^{n+1})_{n \in \mathbb{Z}}$  of objects and morphisms of  $\mathcal{B}$  such that  $d_X^{n+1} d_X^n = 0$ . A complex  $X^\cdot = (X^n, d_X^n : X^n \rightarrow X^{n+1})_{n \in \mathbb{Z}}$  is called bounded below (resp., bounded above, bounded) if  $X^n = 0$  for  $n \ll 0$  (resp.,  $n \gg 0$ ,  $n \ll 0$  and  $n \gg 0$ ).

A complex  $X^\cdot = (X^n, d_X^n)$  is called a stalk complex if there exists an integer  $n_0$  such that  $X^i = 0$  if  $i \neq n_0$ . We identify objects of  $\mathcal{B}$  with a stalk complexes of degree 0.

A morphism  $f : X^\cdot \rightarrow Y^\cdot$  of complexes is a collection of morphisms  $f^n : X^n \rightarrow Y^n$  which makes a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\ & & \downarrow f^n & & \downarrow f^{n+1} & & \\ \cdots & \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

We denote by  $C(\mathcal{B})$  (resp.,  $C^+(\mathcal{B})$ ,  $C^-(\mathcal{B})$ ,  $C^b(\mathcal{B})$ ) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of  $\mathcal{B}$ . An autofunctor  $T : C(\mathcal{B}) \rightarrow C(\mathcal{B})$  is called translation if  $(TX^\cdot)^n = X^{n+1}$  and  $(Td_X)^n = -d_X^{n+1}$  for any complex  $X^\cdot = (X^n, d_X^n)$ .

In  $C(\mathcal{A})$ , a morphism  $u : X^\cdot \rightarrow Y^\cdot$  is called a quasi-isomorphism if  $H_n(u)$  is an isomorphism for any  $n$ .

In this section, “\*” means “nothing”, “+”, “−” or “b”.

**Definition 2.2** For  $u \in \text{Hom}_{C(\mathcal{B})}(X^\cdot, Y^\cdot)$ , the mapping cone of  $u$  is a complex  $M^\cdot(u)$  with

$$\begin{aligned} M^n(u) &= X^{n+1} \oplus Y^n, \\ d_{M^\cdot(u)}^n &= \begin{bmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_Y^n \end{bmatrix} : X^{n+1} \oplus Y^n \rightarrow X^{n+2} \oplus Y^{n+1}. \end{aligned}$$

**Definition 2.3 (Homotopy Relation)** Two morphisms  $f, g \in \text{Hom}_{C(\mathcal{B})}(X^\cdot, Y^\cdot)$  are said to be homotopic (denote by  $f \simeq_h g$ ) if there is a collection of morphisms  $h = (h^n)$ ,  $h^n : X^n \rightarrow Y^{n+1}$  such that  $f^n - g^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$  for all  $n \in \mathbb{Z}$ .

**Definition 2.4 (Homotopy Category)** The homotopy category  $K^*(\mathcal{B})$  of  $\mathcal{B}$  is defined by

1.  $\text{Ob}(K^*(\mathcal{B})) = \text{Ob}(C^*(\mathcal{B}))$ ,
2.  $\text{Hom}_{K^*(\mathcal{B})}(X^\cdot, Y^\cdot) = \text{Hom}_{C^*(\mathcal{B})}(X^\cdot, Y^\cdot) / \simeq_h$  for  $X^\cdot, Y^\cdot \in \text{Ob}(K^*(\mathcal{B}))$ .

**Proposition 2.5** *A category  $K^*(\mathcal{B})$  is a triangulated category whose (distinguished) triangles are defined to be isomorphic to*

$$X^\cdot \xrightarrow{u} Y^\cdot \rightarrow M(u) \rightarrow T(X^\cdot)$$

for any  $u : X^\cdot \rightarrow Y^\cdot$  in  $K^*(\mathcal{B})$ .

**Definition 2.6 (Derived Category)** *The derived category  $D^*(\mathcal{A})$  of an abelian category  $\mathcal{A}$  is  $K^*(\mathcal{A})/K^{*\phi}(\mathcal{A})$ , where  $K^{*\phi}(\mathcal{A})$  is the full subcategory of  $K^*(\mathcal{A})$  consisting of null complexes, that is, complexes whose all cohomologies are 0.*

**Proposition 2.7** *The following hold.*

1.  $D^*(\mathcal{A})$  is a triangulated category, and the canonical functor  $Q : K^*(\mathcal{A}) \rightarrow D^*(\mathcal{A})$  is a  $\partial$ -functor.
2. The  $i$ -th cohomology of complexes is a cohomological functor in the sense of Definition 1.4.

**Proposition 2.8** *If  $0 \rightarrow X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{v} Z^\cdot \rightarrow 0$  is a exact sequence in  $C(\mathcal{A})$ , then it can be embedded in a triangle in  $D(\mathcal{A})$*

$$Q(X^\cdot) \xrightarrow{Q(u)} Q(Y^\cdot) \xrightarrow{Q(v)} Q(Z^\cdot) \rightarrow TQ(X^\cdot).$$

**Definition 2.9 (K-injective Complex)** *A complex  $X^\cdot$  of  $K(\mathcal{B})$  is called K-injective (resp., K-projective) if*

$$\text{Hom}_{K(\mathcal{B})}(N^\cdot, X^\cdot) = 0 \quad (\text{resp., } \text{Hom}_{K(\mathcal{B})}(X^\cdot, N^\cdot) = 0)$$

for any null complex  $N^\cdot$ .

**Example 2.10** *Let  $A$  be a ring,  $\text{Mod } A$  the category of right  $A$ -modules, and  $\text{Inj } A$  (resp.,  $\text{Proj } A$ ) the category of injective (resp., projective) right  $A$ -modules. Then any complex  $I \in K^+(\text{Inj } A)$  (resp.,  $P \in K^-(\text{Proj } A)$ ) is a K-injective (resp., K-projective) complex in  $K(\text{Mod } A)$ .*

**Example 2.11** *Let  $k$  be a field,  $A = k[x]/(x^2)$ , and*

$$X^\cdot : \cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots$$

*Then  $X^\cdot$  is a null complex of finitely generated projective-injective  $A$ -modules. But it is neither K-projective nor K-injective, because  $\text{Hom}_{K(\text{Mod } A)}(X^\cdot, X^\cdot) \neq 0$ .*

**Theorem 2.12 ([Sp], [Ne], [LAM], [Fr])** *Let  $K^{inj}(\text{Mod } A)$  (resp.,  $K^{proj}(\text{Mod } A)$ ) be the category of K-injective (resp., K-projective) complexes, then the following hold.*

1.  $(K^{proj}(\text{Mod } A), K^\phi(\text{Mod } A))$  is a stable  $t$ -structure in  $K(\text{Mod } A)$ , and hence  $D(\text{Mod } A)$  exists and is triangulated equivalent to  $K^{proj}(\text{Mod } A)$ .

2.  $(K^\phi(\text{Mod } A), K^{inj}(\text{Mod } A))$  is a stable  $t$ -structure in  $K(\text{Mod } A)$ , and hence  $D(\text{Mod } A)$  is triangulated equivalent to  $K^{inj}(\text{Mod } A)$ .
3. For a Grothendieck category  $\mathcal{A}$ ,  $(K^\phi(\mathcal{A}), K^{inj}(\mathcal{A}))$  is a stable  $t$ -structure in  $K(\mathcal{A})$ , and hence  $D(\mathcal{A})$  exists and is triangulated equivalent to  $K^{inj}(\mathcal{A})$ .

*Proof.* For a complex  $X^\cdot = (X^i, d^i)$ , we define the following truncations:

$$\begin{aligned}\sigma_{\leq n} X^\cdot &: \cdots \rightarrow X^{n-2} \rightarrow X^{n-1} \rightarrow \text{Ker } d^n \rightarrow 0 \rightarrow \cdots \\ \sigma_{\geq n} X^\cdot &: \cdots \rightarrow 0 \rightarrow \text{Cok } d^{n-1} \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \cdots\end{aligned}$$

(1) For any  $n$ , there is a complex  $P_n^\cdot \in K^-(\text{Proj } A)$  which has a quasi-isomorphism  $P_n^\cdot \rightarrow \sigma_{\leq n} X^\cdot$ . Then we have the following quasi-isomorphisms (qis)

$$X^\cdot \cong \varinjlim \sigma_{\leq n} X^\cdot \xleftarrow{\text{qis}} \varinjlim \sigma_{\leq n} X^\cdot \xleftarrow{\text{qis}} \varinjlim P_n^\cdot$$

Since  $\text{Hom}_C(\coprod_n P_n^\cdot, -) \cong \prod_n \text{Hom}_C(P_n^\cdot, -)$ ,  $\coprod_n P_n^\cdot$  is  $K$ -projective. Here  $h^M = \text{Hom}_C(M, -)$  for any object  $M$ . It is easy to see that  $\varinjlim P_n^\cdot$  is  $K$ -projective by the following exact sequence

$$h \coprod_n P_n^\cdot \rightarrow h \coprod_n P_n^\cdot \rightarrow h \varinjlim P_n^\cdot \rightarrow h^T(\coprod_n P_n^\cdot) \rightarrow h^T(\coprod_n P_n^\cdot)$$

(2) For any  $n$ , there is a complex  $I_n^\cdot \in K^+(\text{Inj } A)$  which has a quasi-isomorphism  $\sigma_{\geq -n} X^\cdot \rightarrow I_n^\cdot$ . Then we have the following quasi-isomorphisms (qis)

$$X^\cdot \cong \varprojlim \sigma_{\geq -n} X^\cdot \xrightarrow{\text{qis}} \varprojlim \sigma_{\geq -n} X^\cdot \xrightarrow{\text{qis}} \varprojlim I_n^\cdot$$

by the same reason of (1), we have the statement.

(3) Because there is a ring  $A$  such that  $\mathcal{A}$  is a localization of  $\text{Mod } A$  (Gabriel-Popescu Theorem). See [LAM] or [Fr] for details.  $\square$

**Remark 2.13** If  $P^\cdot$  is a  $K$ -projective complex (e.g. a bounded above complex of projective  $A$ -modules), then we have

$$\text{Hom}_{K(\text{Mod } A)}(P^\cdot, X^\cdot) \cong \text{Hom}_{D(\text{Mod } A)}(P^\cdot, X^\cdot)$$

for any complex  $X^\cdot$ . Similarly, for a  $K$ -injective complex  $I^\cdot$  (e.g. bounded below complex of injective  $A$ -modules), then we have

$$\text{Hom}_{K(\text{Mod } A)}(X^\cdot, I^\cdot) \cong \text{Hom}_{D(\text{Mod } A)}(X^\cdot, I^\cdot)$$

for any complex  $X^\cdot$ . In particular, given  $A$ -modules  $M, N$ , we have

$$\text{Ext}_A^i(M, N) \cong \text{Hom}_{D(\text{Mod } A)}(M, N[i])$$

**Definition 2.14 (Double Complex, Total complex)** A double complex  $C^{\bullet,\bullet}$  is a bi-graded object  $(C^{p,q})_{p,q \in \mathbb{Z}}$  of  $\mathcal{A}$  together with  $d_I^{p,q} : C^{p,q} \rightarrow C^{p+1,q}$  and  $d_{II}^{p,q} : C^{p,q} \rightarrow C^{p,q+1}$  such that

$$C^{\bullet,q} = (C^{p,q}, d_I^{p,q} : C^{p,q} \rightarrow C^{p+1,q}), \quad C^{p,\bullet} = (C^{p,q}, d_{II}^{p,q} : C^{p,q} \rightarrow C^{p,q+1})$$

are complexes satisfying  $d_I^{p,q+1} d_{II}^{p,q} - d_{II}^{p+1,q} d_I^{p,q} = 0$ . For a double complex  $C^{\bullet,\bullet}$ , we define the total complexes

$$\begin{aligned} \text{Tot}^{\text{II}} C^{\bullet,\bullet} &= (X^n, d^n), \text{ where } X^n = \coprod_{p+q=n} C^{p,q}, d^n = \coprod_{p+q=n} (d_I^{p,q} + (-1)^p d_{II}^{p,q}) \\ \text{Tot}^{\text{I}} C^{\bullet,\bullet} &= (Y^n, d^n), \text{ where } Y^n = \prod_{p+q=n} C^{p,q}, d^n = \prod_{p+q=n} (d_I^{p,q} + (-1)^p d_{II}^{p,q}). \end{aligned}$$

**Definition 2.15 (Cartan-Eilenberg Resolution)** For a complex  $X^\bullet \in \text{D}(\text{Mod } A)$ , let

$$\cdots \rightarrow P^{-1\bullet} \rightarrow P^{0\bullet} \rightarrow X^\bullet \rightarrow 0 \quad (\text{resp., } 0 \rightarrow X^\bullet \rightarrow I^{0\bullet} \rightarrow I^{1\bullet} \rightarrow \cdots)$$

be an exact sequence with  $P^n$  (resp.,  $I^n$ ) being a complex of projective (resp., injective)  $A$ -modules. We call  $\cdots \rightarrow P^{-1\bullet} \rightarrow P^{0\bullet}$  (resp.,  $I^{0\bullet} \rightarrow I^{1\bullet} \rightarrow \cdots$ ) a Cartan-Eilenberg projective (resp., injective) resolution of  $X^\bullet$  if the induced complexes  $\cdots \rightarrow B^n(P^{-1\bullet}) \rightarrow B^n(P^{0\bullet})$  and  $\cdots \rightarrow H^n(P^{-1\bullet}) \rightarrow H^n(P^{0\bullet})$  (resp.,  $B^n(I^{0\bullet}) \rightarrow B^n(I^{1\bullet}) \rightarrow \cdots$  and  $H^n(I^{0\bullet}) \rightarrow H^n(I^{1\bullet}) \rightarrow \cdots$ ) are also projective (resp., injective) resolutions of  $B^n(X^\bullet), H^n(X^\bullet)$ , respectively.

**Proposition 2.16** Under the setting of Definition 2.15, the following hold.

1.  $\text{Tot}^{\text{II}} P^{\bullet,\bullet}$  is  $K$ -projective, and the induced morphism of complexes  $\text{Tot}^{\text{II}} P^{\bullet,\bullet} \rightarrow X^\bullet$  is a quasi-isomorphism.
2.  $\text{Tot}^{\text{I}} I^{\bullet,\bullet}$  is  $K$ -injective, and the induced morphism of complexes  $X^\bullet \rightarrow \text{Tot}^{\text{I}} I^{\bullet,\bullet}$  is a quasi-isomorphism.

*Sketch of Proof.* We consider the following truncations

$$\sigma_{\leq n}^{\text{II}} P^{\bullet,\bullet} : \cdots \rightarrow \sigma_{\leq n} P^{-1\bullet} \rightarrow \sigma_{\leq n} P^{0\bullet}, \quad \sigma_{\geq n}^{\text{II}} I^{\bullet,\bullet} : \sigma_{\geq n} I^{0\bullet} \rightarrow \sigma_{\geq n} I^{1\bullet} \rightarrow \cdots$$

Then it is easy to see  $\text{Tot}^{\text{II}} \sigma_{\leq n}^{\text{II}} P^{\bullet,\bullet}$  (resp.,  $\text{Tot}^{\text{I}} \sigma_{\geq n}^{\text{II}} I^{\bullet,\bullet}$ ) is  $K$ -projective (resp.,  $K$ -injective), and that the induced morphism of complexes  $\text{Tot}^{\text{II}} \sigma_{\leq n}^{\text{II}} P^{\bullet,\bullet} \rightarrow \sigma_{\leq n} X^\bullet$  (resp.,  $\sigma_{\geq n} X^\bullet \rightarrow \text{Tot}^{\text{I}} \sigma_{\geq n}^{\text{II}} I^{\bullet,\bullet}$ ) is a quasi-isomorphism. Therefore we have the following quasi-isomorphisms (qis)

$$\begin{aligned} X^\bullet &\xleftarrow{\text{qis}} \varinjlim \sigma_{\leq n} X^\bullet \xleftarrow{\text{qis}} \varinjlim \text{Tot}^{\text{II}} \sigma_{\leq n}^{\text{II}} P^{\bullet,\bullet} \xrightarrow{\text{qis}} \text{Tot}^{\text{II}} P^{\bullet,\bullet} \\ (\text{resp., } X^\bullet &\xrightarrow{\text{qis}} \varprojlim \sigma_{\geq -n} X^\bullet \xrightarrow{\text{qis}} \varprojlim \text{Tot}^{\text{I}} \sigma_{\geq -n}^{\text{II}} I^{\bullet,\bullet} \xleftarrow{\text{qis}} \text{Tot}^{\text{I}} I^{\bullet,\bullet}) \end{aligned}$$

and  $\text{Tot}^{\text{II}} P^{\bullet,\bullet}$  (resp.,  $\text{Tot}^{\text{I}} I^{\bullet,\bullet}$ ) is  $K$ -projective (resp.,  $K$ -injective).  $\square$

**Definition 2.17 (Right Derived Functor)** For a  $\partial$ -functor  $F : K^*(\mathcal{A}) \rightarrow K(\mathcal{A}')$ , the right derived functor of  $F$  is a  $\partial$ -functor

$$R^*F : D^*(\mathcal{A}) \rightarrow D(\mathcal{A}')$$

together with a functorial morphism of  $\partial$ -functors

$$\xi \in \partial \text{Mor}(Q_{\mathcal{A}'} \circ F, R^*F \circ Q_{\mathcal{A}}^*)$$

with the following property:

For  $G \in \partial(D^*(\mathcal{A}), D(\mathcal{A}'))$  and  $\zeta \in \partial \text{Mor}(Q_{\mathcal{A}'} \circ F, G \circ Q_{\mathcal{A}}^*)$ , there exists a unique morphism  $\eta \in \partial \text{Mor}(R^*F, G)$  such that

$$\zeta = (\eta Q_{\mathcal{A}}^*)\xi.$$

In other words, we can simply write the above using functor categories. For triangulated categories  $\mathcal{C}, \mathcal{C}'$ , the  $\partial$ -functor category  $\partial(\mathcal{C}, \mathcal{C}')$  is the category (?) consisting of  $\partial$ -functors from  $\mathcal{C}$  to  $\mathcal{C}'$  as objects and  $\partial$ -functorial morphisms as morphisms. Then we have

$$\begin{array}{ccc} \partial \text{Mor}(Q_{\mathcal{A}'} \circ F, -Q_{\mathcal{A}}^*) \cong \partial \text{Mor}(R^*F, -) & K^*(\mathcal{A}) & \xrightarrow{F} K(\mathcal{A}') \\ & \downarrow Q_{\mathcal{A}} & \downarrow Q_{\mathcal{A}'} \\ & D^*(\mathcal{A}) & \xrightarrow[\underset{G}{R^*F}]{R^*F} D(\mathcal{A}') \end{array}$$

as functors from  $\partial(D^*(\mathcal{A}), D(\mathcal{A}'))$  to  $\mathfrak{Set}$ .

**Proposition 2.18** Let  $\mathcal{A}, \mathcal{A}'$  be abelian categories,  $F : K(\mathcal{A}) \rightarrow K(\mathcal{A}')$  a  $\partial$ -functor. If  $\mathcal{A}$  is a Grothendieck category, then we have the right derived functor  $R^*F : D(\mathcal{A}) \rightarrow D(\mathcal{A}')$  such that  $F(X^\cdot) \cong R^*F(X^\cdot)$  for any  $K$ -injective complex  $X^\cdot$ .

**Remark 2.19** In the setting of Definition 2.17, the left derived functor  $L^*F : D^*(\mathcal{A}) \rightarrow D(\mathcal{A}')$  can be also defined by reversing arrows of  $\partial$ -functorial morphisms. Let  $R^n F(X^\cdot) = H^n(R^*F(X^\cdot))$ ,  $L^n F(X^\cdot) = H^n(L^*F(X^\cdot))$ , then  $R^n F$  (resp.,  $L^n F$ ) coincides with the ordinary definition of the  $n$ -th right (resp., left) derived functor. According to Proposition 2.16, if  $F$  commutes with products (resp., coproducts), then the  $n$ -th hypercohomology  $R^n F$  (resp., hyperhomology  $L^n F$ ) coincides with  $R^n F$  (resp.,  $L^n F$ ) (c.f. [CE], [Mc], [We]).

**Definition 2.20** ( $\text{Hom}_A^\cdot, \otimes_A$ ) Let  $X^\cdot, Y^\cdot$  be complexes in  $C(\text{Mod } A)$ ,  $Z^\cdot$  a complex in  $C(\text{Mod } A^{\text{op}})$ . We define the complex  $\text{Hom}_A^\cdot(X^\cdot, Y^\cdot)$  in  $C(\mathfrak{Ab})$  by

$$\text{Hom}_A^n(X^\cdot, Y^\cdot) = \prod_{j-i=n} \text{Hom}_A(X^i, Y^j), \quad d_{\text{Hom}_A^\cdot(X,Y)}^n(f) = d_X \circ f - (-1)^n f \circ d_Y$$

for  $f \in \text{Hom}_A^n(X^\cdot, Y^\cdot)$ . And we define the complex  $X^\cdot \otimes_A Z^\cdot$  in  $C(\mathfrak{Ab})$  by

$$X^\cdot \otimes_A^n Z^\cdot = \prod_{i+j=n} X^i \otimes_A Z^j, \quad d_{X \otimes Y}^n = d_X \otimes 1 + (-1)^n 1 \otimes d_Z$$

**Proposition 2.21** *Let  $A$  be a ring. Then we have a right derived functor*

$$\mathbf{R} \operatorname{Hom}_A^\bullet : D(\operatorname{Mod} A)^{op} \times D(\operatorname{Mod} A) \rightarrow D(\mathfrak{Ab})$$

*and a left derived functor*

$$\dot{\otimes}_A^L : D(\operatorname{Mod} A) \times D(\operatorname{Mod} A^{op}) \rightarrow D(\mathfrak{Ab})$$

**Proposition 2.22** *Let  $A$  be a ring. For complexes  $X^\bullet, Y^\bullet$ , we have isomorphisms*

$$\begin{aligned} H^n(\operatorname{Hom}_A^\bullet(X^\bullet, Y^\bullet)) &\cong \operatorname{Hom}_{K(\operatorname{Mod} A)}(X^\bullet, Y^\bullet[n]) \\ H^n(\mathbf{R} \operatorname{Hom}_A^\bullet(X^\bullet, Y^\bullet)) &\cong \operatorname{Hom}_{D(\operatorname{Mod} A)}(X^\bullet, Y^\bullet[n]) \end{aligned}$$

**Definition 2.23 (Perfect Complex)** *Let  $A$  be a ring. A complex  $X^\bullet \in D(\operatorname{Mod} A)$  is called a perfect complex if  $X^\bullet$  is quasi-isomorphic to a bounded complex of finitely generated projective  $A$ -modules.*

*Let  $X$  be a scheme,  $D(X)$  the derived category of sheaves of  $\mathcal{O}_X$ -modules. We denote by  $D_{qc}(X)$  the full subcategory of  $D(X)$  consisting of complexes whose cohomologies are quasi-coherent sheaves. A complex  $X^\bullet \in D_{qc}(X)$  is called a perfect complex if  $X^\bullet$  is locally quasi-isomorphic to a bounded complex of vector bundles (See [TT]).*

*We denote by  $D_{pf}(\mathcal{A})$  the full triangulated subcategory of  $D(\mathcal{A})$  consisting of perfect complexes.*

**Proposition 2.24** ([Rd1], [Ne]) *For a ring  $A$ , the following hold.*

1. *A complex  $X^\bullet \in D(\operatorname{Mod} A)$  is perfect if and only if it is a compact object in  $D(\operatorname{Mod} A)$ .*
2.  *$D(\operatorname{Mod} A)$  is compactly generated.*

**Theorem 2.25** ([BV]) *Let  $X$  be a quasi-compact quasi-separated scheme, then the following hold.*

1. *A complex  $X^\bullet \in D_{qc}(X)$  is perfect if and only if it is a compact object in  $D_{qc}(X)$ .*
2.  *$D_{qc}(X)$  is compactly generated.*

**Theorem 2.26** ([BN]) *Let  $X$  be a quasi-compact separated scheme, then the canonical functor  $D(\operatorname{Qcoh} X) \rightarrow D_{qc}(X)$  is a triangulated equivalence, where  $\operatorname{Qcoh} X$  is the category of quasi-coherent sheaves of  $\mathcal{O}_X$ -modules.*

**Corollary 2.27** ([BV]) *Let  $X$  be smooth over a field, then we have*

$$D^b(\operatorname{coh} X) \stackrel{\Delta}{\cong} D_{pf}(X).$$

*where  $\operatorname{coh} X$  is the category of coherent sheaves of  $\mathcal{O}_X$ -modules.*

For a ring  $A$ , we denote by  $\text{proj } A$  the category of finitely generated projective  $A$ -modules.

**Theorem 2.28** ([Rd1], [Rd2]) *Let  $A, B$  be algebras over a field  $k$ . The following are equivalent.*

1.  $D(\text{Mod } A) \xrightarrow{\Delta} D(\text{Mod } B)$ .
2.  $K^b(\text{proj } A) \xrightarrow{\Delta} K^b(\text{proj } B)$ .
3. *There is a perfect complex  $T^\bullet \in D(\text{Mod } A)$  such that*
  - (a)  $B \cong \text{End}_{D(\text{Mod } A)}(T^\bullet)$ ,
  - (b)  $\text{Hom}_{D(\text{Mod } A)}(T^\bullet, T^\bullet[i]) = 0$  for  $i \neq 0$ ,
  - (c)  $\{T^\bullet[i] \mid i \in \mathbb{Z}\}$  *is a generating set in*  $D(\text{Mod } A)$ .
4. *There is a complex  $V^\bullet$  of  $B$ - $A$ -bimodules such that*

$$R\text{Hom}_A(V^\bullet, -) : D(\text{Mod } A) \rightarrow D(\text{Mod } B)$$

*is an equivalence.*

*In this case,  $T^\bullet$  is called a tilting complex for  $A$ ,  $V^\bullet$  is called a two-sided tilting complex, and  $R\text{Hom}_A(V^\bullet, -)$  is called a standard equivalence.*

**Theorem 2.29** ([BO]) *Let  $X$  be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. If  $X'$  is a smooth algebraic variety such that  $D^b(\text{coh } X) \xrightarrow{\Delta} D^b(\text{coh } X')$ , then  $X'$  is isomorphic to  $X$ .*

**Theorem 2.30** ([Be]) *Let  $\mathbf{P} = \mathbf{P}_k^n$  be the  $n$ -dimensional projective space over a field  $k$ , and let  $T_1 = \bigoplus_{i=0}^n \mathcal{O}(i)$ ,  $T_2 = \bigoplus_{i=0}^n \Omega(-i)$ , and  $B_1 = \text{End}_{\mathbf{P}}(T_1)$ ,  $B_2 = \text{End}_{\mathbf{P}}(T_2)$ . Then  $B_i$  is a finite dimensional  $k$ -algebra of finite global dimension, and*

$$D^b(\text{coh } \mathbf{P}) \xrightarrow{\Delta} D^b(\text{mod } B_i)$$

*where  $\text{mod } B_i$  is the category of finitely generated  $B_i$ -modules ( $i = 1, 2$ ).*

**Definition 2.31** *Let  $A$  be an algebra over a field  $k$ . The derived Picard group of  $A$  (relative to  $k$ ) is*

$$\text{DPic}_k(A) := \frac{\{\text{two-sided tilting complexes } T \in D^b(\text{Mod } A^e)\}}{\text{isomorphism}}$$

*with identity element  $A$ , product  $(T_1, T_2) \mapsto T_1 \otimes_A^L T_2$  and inverse  $T \mapsto T^\vee := R\text{Hom}_A(T, A)$ . Given any  $k$ -linear triangulated category  $\mathcal{C}$  we let*

$$\text{Out}_k^\Delta(\mathcal{C}) := \frac{\{k\text{-linear triangulated self-equivalences of } \mathcal{C}\}}{\text{isomorphism}}.$$



**Theorem 2.32** ([MY]) *Let  $k$  be an algebraically closed field, and  $A$  a finite dimensional hereditary  $k$ -algebra. Then we have*

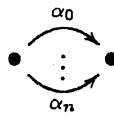
$$\mathrm{DPic}_k(A) = \mathrm{Out}_k^\Delta(\mathrm{D}^b(\mathrm{Mod} A)) = \mathrm{Out}_k^\Delta(\mathrm{D}^b(\mathrm{mod} A))$$

M. Kontsevich and A. Rosenberg introduced the notion of non-commutative projective spaces  $\mathbf{NP}^n$  [KR], and showed that

$$\mathrm{D}^b(\mathrm{Qcoh} \mathbf{NP}^n) \stackrel{\Delta}{\cong} \mathrm{D}^b(\mathrm{Mod} kQ_n)$$

$$\mathrm{D}^b(\mathrm{coh} \mathbf{NP}^n) \stackrel{\Delta}{\cong} \mathrm{D}^b(\mathrm{mod} kQ_n)$$

where  $Q_n$  is the quiver



**Corollary 2.33** ([MY]) *For non-commutative projective spaces  $\mathbf{NP}^n$ , we have*

$$\begin{aligned} \mathrm{Out}_k^\Delta(\mathrm{D}^b(\mathrm{Qcoh} \mathbf{NP}^n)) &\cong \mathrm{Out}_k^\Delta(\mathrm{D}^b(\mathrm{coh} \mathbf{NP}^n)) \\ &\cong \mathbb{Z} \times (\mathbb{Z} \ltimes \mathrm{PGL}_{n+1}(k)) \end{aligned}$$

**Theorem 2.34** ([BO]) *Let  $X$  be a smooth irreducible projective variety with ample canonical or anticanonical sheaf. Then  $\mathrm{Out}_k^\Delta(\mathrm{D}^b(\mathrm{coh} X))$  is generated by the automorphisms of variety, the twists by invertible sheaves and the translations, and hence  $\mathrm{Out}_k^\Delta(\mathrm{D}^b(\mathrm{coh} X)) \cong (\mathrm{Aut}_k X \ltimes \mathrm{Pic} X) \times \mathbb{Z}$ .*

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